MASTER IN **RENEWABLE ENERGY IN THE MARINE** ENVIRONMENT

REM

# Simplifying complexity using dimensional analysis

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# Wave tank/flume tests

# What are the conditions for hydrodynamic similarity?

# Dimensions versus Units

- The dimension of a physical quantity is independent of the units of measure
- The physical quantities are measured in units

**Fundamental dimensions**

| Dimension | Symbol | Unit SI | Name |
|---|---|---|---|
| Mass | M | kg | kilogram |
| Length | L | m | metre |
| Time | T | s | second |
| Temperature | $\Theta$ | K | Kelvin |
| Electrical current | I | A | Ampere |
| Amount of substance | N | mol | mole |
| Luminous intensity | J | cd | candela |
| Electric charge | e | C | Coulomb |

# Dimensions versus Units

- **Notation:** square brackets around some physical quantity $q$ to denote its dimension
- Example: Dimension of velocity $[v] = LT^{-1}$
- Same physical quantities are dimensionless, for example, angles
- In the SI system, angles, $\theta$, are the ratio between the arc length $s$ and the radius $r$

$$\theta = \frac{s}{r}$$

- As such

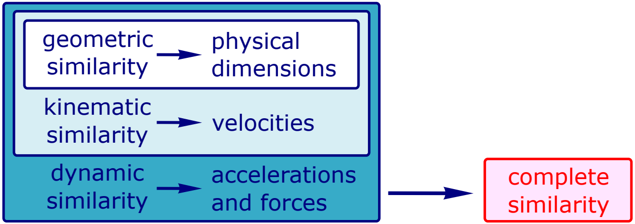
$$[\theta] = 1$$

# The concept of similarity

- In dimensional analysis
  - **Prototype** - real scale or real physical system
  - **model** - system at laboratory scale
- Usually the model is smaller than the prototype
- The model does need to be smaller
- Model and prototype can have similar dimensions and diferent velocities, fluid properties, etc

# The concept of similarity

- Two experiments are similar if the physical phenomena are equal for both
- Two system operating in different conditions are described by the same set of equations, i.e., they are similar, if exists a scale ratio between certain physical quantities of both systems in all domain




- Complete similarity only exists in simple engineering problems

# Buckingham's or Π theorem

- In 1914 Buckingham stated the following theorem

A dimensionally homogeneous equation

$$x_1 = f\left(x_2, \ldots, x_n\right)$$

can be reduced to a relationship of $m = n - p$ dimensionless groups called Πs,

$$\Pi_1 = \phi\left(\Pi_2, \ldots, \Pi_m\right)$$

where $p$ is the number of dimensions of the sub-set of $p$ primitive variables $\{x_1, \ldots, x_p\}$

# Buckingham's or Π theorem

- We can define dimensionless groups $\Pi_j$ provided that
  - The set of Πs has $m$ dimensionless groups $\{\Pi_1, \ldots, \Pi_m\}$
  - The set of Πs includes all primitive variables $\{x_1, \ldots, x_n\}$
  - The Π groups are independent
    - For example, we cannot define

      $$\Pi_1 = \frac{x_1}{x_2}$$

      and

      $$\Pi_2 = \frac{x_2}{x_1}$$

      since

      $$\Pi_2 = \Pi_1^{-1}$$

# Buckingham's or Π theorem



- **Flow around a bi-dimensional circular cylinder**

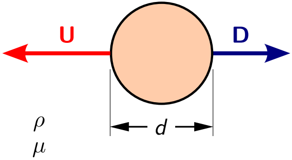
- Drag, $D$, is a function of: velocity $U$, diameter, $d$, $\epsilon$ relative roughness, and fluid properties, density, $\rho$, and viscosity, $\mu$

$$D = f(U,d,\epsilon,\rho,\mu) = 0$$

- Dimensions

$$[D] = \frac{ML}{T^2} \quad [U] = \frac{L}{T} \quad [d] = L$$

$$[\epsilon] = L \quad [\rho] = \frac{M}{L^3} \quad [\mu] = \frac{M}{LT}$$




- Summarizing

$$\left.\begin{array}{l} n = 6 \text{ variables} \\ p = 3 \text{ dimensions (M,L,T)} \end{array}\right\} \Rightarrow m = 6-3 = 2 \text{ dimensionless groups } \{\Pi_1,\Pi_2\}$$

# Buckingham's or Π theorem

## Example 1

- Let us select three primitives variables involving the three dimensions LMT
- For example: $U$, $D$ and $\rho$

$$[U] = LT^{-1}$$

$$[d] = L$$

$$[\rho] = ML^{-3}$$

- Define two dimensionless groups using the remaining primitive variables

$$\Pi_1 = \frac{D}{\rho U^2 d^2} \quad \Rightarrow \quad C_{\mathrm{D}} = \frac{D}{\frac{1}{2}\rho U^2 d^2} \quad \rightarrow \quad \textbf{Drag coefficient}$$

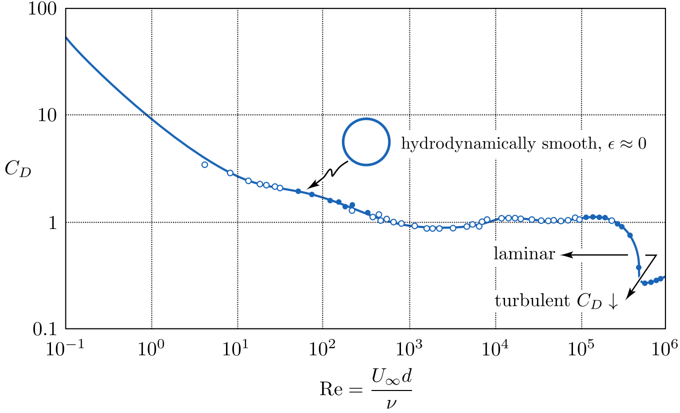
$$\Pi_2 = \frac{\mu}{\rho U d} \quad \Rightarrow \quad \mathrm{Re} = \frac{\rho U d}{\mu} \quad \rightarrow \quad \textbf{Reynolds number}$$

$$\Pi_3 = \frac{\epsilon}{d} \quad \Rightarrow \quad \textbf{Relative roughness}$$

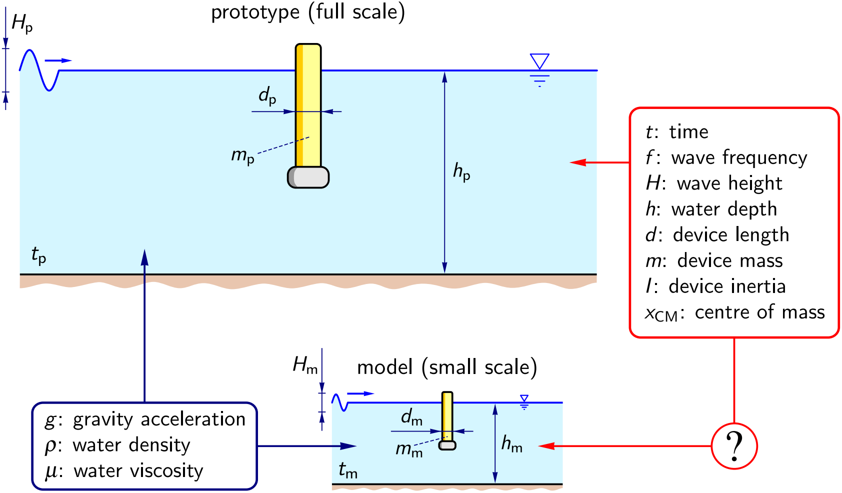
# Buckingham's or Π theorem

## Example 1

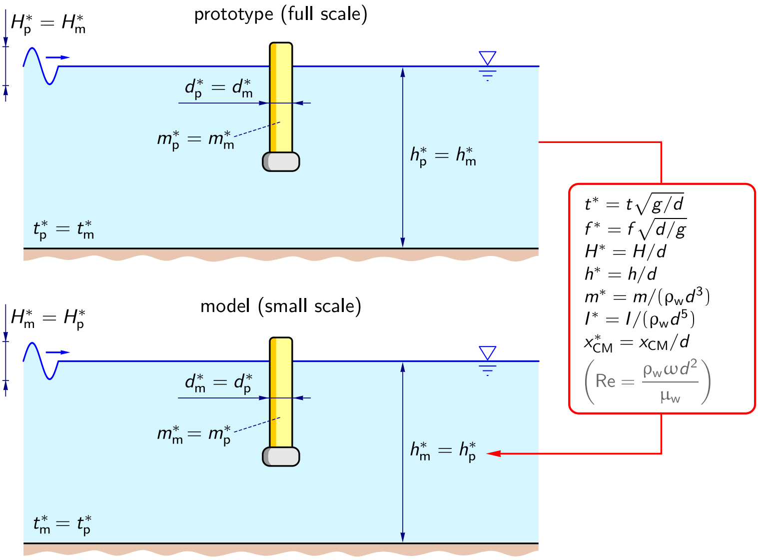
- The Π theorem does not allow us to know function $\phi$ but allows us to understand the relevant variables for the problem
- Instead of performing experimental for all combinations $U$, $d$, $\rho$ and $\mu$, we just need a small set of experiments to determine $C_D$ as a function of Re



# What are the conditions for hydrodynamic similarity?



# What are the conditions for hydrodynamic similarity?




In dimensionless variables prototype and model response should be equal

# Testing conditions model/prototype

**Independent variables:**

- $t$: time
- $f$: wave frequency
- $H$: wave height
- $h$: water depth
- $d$: characteristic length of the device
- $m$: mass of device
- $I$: inertia of device
- $x_{\mathrm{CM}}$: centre of mass
- $g$: gravity acceleration
- $\rho$: water density
- $\mu$: water viscosity

**Dependent variables:**

- $x$: displacement of device
- $\dot{x}$: velocity of device
- $\ddot{x}$: acceleration of device
- $\dot{\theta}$: angular speed of device
- $\ddot{\theta}$: angular acceleration of device
- $p$: pressure on device
- $F$: force on device
- $T$: torque on device
- $P$: power on device

$$\left\{t, x, \dot{x}, \ddot{x}, \dot{\theta}, \ddot{\theta}, p, F, T, P\right\} = \text{function}\Big(f, H, h, d, m, I, x_{\mathrm{CM}}, g, \rho, \mu\Big)$$

# Buckingham's Π theorem

- The number of independent variables reduces by the number of fundamental dimensions
- The number of fundamental dimensions is 3 for this problem [LMT]
- Select 3 primary variables (**choice not unique**!)
- Common choice
  - diameter the device $[d] = L$
  - acceleration of gravity $[g] = LT^{-2}$
  - water density $[\rho] = ML^{-3}$
- Alternative choice
  - water depth $[h] = L$
  - wave frequency $[f] = T^{-1}$
  - Water density $[\rho] = ML^{-3}$

# Buckingham's Π theorem with $d$, $g$ and $\rho$ as primary variables

**Dimensionless independent variables:**

- $t^* = t\sqrt{g/d}$
- $f^* = f\sqrt{d/g}$
- $H^* = H/d$
- $h^* = h/d$
- $m^* = m/(\rho d^3)$
- $I^* = I/(\rho d^5)$
- $x^*_{\text{CM}} = x_{\text{CM}}/d$
- $\text{Re} = \dfrac{\rho g^{1/2} d^{3/2}}{\mu}$: **Reynolds number**
- The number of independent variables was reduced by 3

**Dimensionless dependent variables:**

- $x^* = x/d$
- $\text{Fr} = \dot{x}^* = \dot{x}/\sqrt{gd}$: **Froude number**
- $\ddot{x}^* = \ddot{x}/g$
- $\dot{\theta}^* = \dot{\theta}\sqrt{d/g}$
- $\ddot{\theta}^* = \ddot{\theta}\, d/g$
- $p^* = p/(\rho g d)$
- $F^* = F/(\rho g d^3)$
- $T^* = T/(\rho g d^4)$
- $P^* = P/(\rho g^{3/2} d^{7/2})$

$$\left\{x^*, \text{Fr}, \ddot{x}^*, \dot{\theta}^*, \ddot{\theta}^*, p^*, F^*, T^*, P^*\right\} = \text{function}\left(t^*, f^*, H^*, h^*, m^*, I^*, y^*_{\text{CM}}, \text{Re}\right)$$

# Buckingham's Π theorem with $d$, $g$ and $\rho$ as primary variables

- To have the same response, **the dimensionless numbers of the prototype scale and the model scale must be the same**

$$\left\{x^*, \mathrm{Fr}, \ddot{x}^*, \theta^*, \dot{\theta}^*, \ddot{\theta}^*, p^*, F^*, T^*, P^*\right\} = \text{function}\left(t^*, f^*, H^*, h^*, m^*, I^*, y^*_{\mathrm{CM}}, \mathrm{Re}\right)$$

- All independent dimensionless variables scale with the geometry except
  - $t^*$, Fr and Re
- $t^*$ and Fr are both the time scale
- The Re is ratio between the inertia forces and the viscous forces
  - The effect of the Reynolds number can be understood through the Navier-Stokes equations

# How to easily obtain Froude scaling factors

- Units appearing in the Froude scale $M$, $L$ and $T$
- In the International System of Units (SI)
  - $[M] = \text{kg}$
  - $[L] = \text{m}$
  - $[T] = \text{s}$
- Chose 3 constants that combine the 3 units

$$\left.\begin{array}{rcccccc} \text{Acceleration of gravity:} & g & \Rightarrow & [M/T^2] & = & \text{m/s}^2 \\ \text{Characteristic length:} & d & \Rightarrow & [L] & = & \text{m} \\ \text{Water density:} & \rho_w & \Rightarrow & [M/L^3] & = & \text{kg/m}^3 \end{array}\right\} \Rightarrow \{M,\ L,\ T\}$$

- Example 1: Dimensionless time

$$t^* = t\sqrt{\frac{g}{d}}$$

- Example 2: Dimensionless velocity

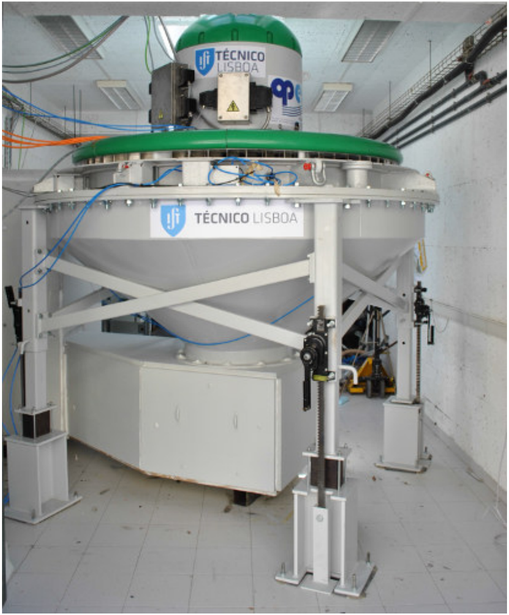
$$u^* = \frac{u}{\sqrt{gd}}$$

## Froude scaling factors used for scaled model testing

| Property | Dimensions | Scaling factor |
|---|---|---|
| Length | $L$ | $\varepsilon$ |
| Mass | $M$ | $(\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon^3$ |
| Time | $T$ | $\varepsilon^{1/2}$ |
| Wave height | $L$ | $\varepsilon$ |
| Water depth | $L$ | $\varepsilon$ |
| Displacement | $L$ | $\varepsilon$ |
| Velocity | $LT^{-1}$ | $\varepsilon^{1/2}$ |
| Acceleration | $LT^{-2}$ | 1 |
| Wave period | $T$ | $\varepsilon^{1/2}$ |
| Wave frequency | $T^{-1}$ | $\varepsilon^{-1/2}$ |
| Moment of inertia | $ML^2$ | $(\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon^5$ |
| Moment of area | $L^4$ | $\varepsilon^4$ |
| Force | $MLT^{-2}$ | $(\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon^3$ |
| Moment | $ML^2T^{-2}$ | $(\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon^4$ |
| Pressure | $ML^{-1}T^{-2}$ | $(\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon$ |
| Power | $ML^2T^{-3}$ | $(\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon^{7/2}$ |
| Angle | – | 1 |

- $\varepsilon = d_m/d_p$ (model/prototype scale)
- All scales of the physical quantities can be obtained from the basic $M$, $L$ and $T$
  - Pressure: $ML^{-1}T^{-2} \Rightarrow \left[(\rho_\mathrm{p}/\rho_\mathrm{m})\varepsilon^3\right][\varepsilon]^{-1}\left[\varepsilon^{1/2}\right]^{-2} \Rightarrow (\rho_\mathrm{p}/\rho_\mathrm{m})\,\varepsilon$

# What are the conditions for turbine testing similarity?



# Dimensional analysis applied to turbines

- Determine the performance of a turbine independently from the size and fluid properties (**assuming incompressible flow**)

$$\begin{aligned} \text{mass flow rate:} \quad & \dot{m} = f_1\left(\Delta p, \Omega, d, \rho, \mu, \epsilon\right) \\ \text{Torque:} \quad & T = f_2\left(\Delta p, \Omega, d, \rho, \mu, \epsilon\right) \\ \text{Power:} \quad & P = f_3\left(\Delta p, \Omega, d, \rho, \mu, \epsilon\right) \\ \text{Efficiency:} \quad & \eta = f_4\left(\Delta p, \Omega, d, \rho, \mu, \epsilon\right) \end{aligned}$$

- Buckingham's theorem of dimensional analysis allows us to reduce the number of independent variables from six to three, the number of fundamental dimensions [LMT]

- Let us choose three variables such that all fundamental dimensions [LMT] appear: $[\rho] = \mathrm{ML}^{-3}$, $[d] = \mathrm{L}$, $[\Omega] = \mathrm{T}^{-1}$

# Dimensional analysis applied to turbines

- Using a combination of $\rho$, $d$ and $\Omega$ we can the get dimensionaless values

$$\frac{\dot{m}}{\rho\Omega d^3} = f_1\left(\frac{\Delta p}{\rho\Omega^2 d^2}, \frac{\rho\Omega d^2}{\mu}, \frac{\epsilon}{d}\right)$$

$$\frac{T}{\rho\Omega^2 d^5} = f_2\left(\frac{\Delta p}{\rho\Omega^2 d^2}, \frac{\rho\Omega d^2}{\mu}, \frac{\epsilon}{d}\right)$$

$$\frac{P}{\rho\Omega^3 d^5} = f_3\left(\frac{\Delta p}{\rho\Omega^2 d^2}, \frac{\rho\Omega d^2}{\mu}, \frac{\epsilon}{d}\right)$$

$$\eta = f_4\left(\frac{\Delta p}{\rho\Omega^2 d^2}, \frac{\rho\Omega d^2}{\mu}, \frac{\epsilon}{d}\right)$$

$$\mathrm{Re} = \frac{\rho\Omega d^2}{\mu}$$

$$\Psi = \frac{\Delta p}{\rho\Omega^2 d^2}$$

$$\Phi = f_\Psi\left(\Psi, \mathrm{Re}, \frac{\epsilon}{d}\right)$$

$$\Pi = f_\Pi\left(\Psi, \mathrm{Re}, \frac{\epsilon}{d}\right)$$

$$\eta = f_\eta\left(\Psi, \mathrm{Re}, \frac{\epsilon}{d}\right)$$

- We reduce the number of variables of the original problem from 6 to 3 ([LMT])
- The turbine efficiency $\eta$ is already dimensionless
- Torque and Power imply the same dimensionless number $\Pi$
- **NOTE**: for turbomachines we don't use **Froude scale** $\Rightarrow$ different physics

# Dimensional analysis applied to turbines

- If we characterize the turbine under fully developed turbulence conditions and considering the same relative roughness

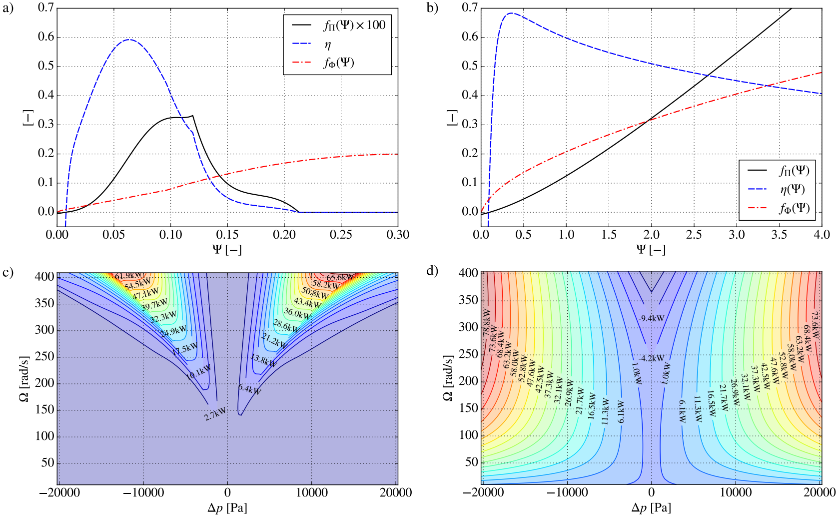
$$\Phi = f_\Psi(\Psi)$$

$$\Pi = f_\Pi(\Psi)$$

$$\eta = f_\eta(\Psi)$$

- Typically the curves are independent of the Reynolds number for $\mathrm{Re} > 10^6$ for small Mach numbers $\mathrm{Ma} = \dfrac{\Omega d}{2a} < 0.3$ where $a$ is the speed of sound

# Dimensional analysis applied to turbines




**The operating range of a Wells turb. is much smaller than a biradial turb.**

# How to scale a turbine

- Pressure on the OWC scales with Froude (neglecting water density differences)

$$p_\mathrm{p}^* = p_\mathrm{m}^* \;\Rightarrow\; \frac{\Delta p_\mathrm{p}}{\rho_\mathrm{p} g L_\mathrm{p}} = \frac{\Delta p_\mathrm{m}}{\rho_\mathrm{m} g L_\mathrm{m}} \;\Rightarrow\; \frac{\Delta p_\mathrm{m}}{\Delta p_\mathrm{p}} = \frac{L_\mathrm{m}}{L_\mathrm{p}} = \varepsilon \tag{1}$$

  where $L$ is the characteristic length of the buoy

- Flow rate on the OWC scales with Froude

$$Q_\mathrm{p}^* = Q_\mathrm{m}^* \;\Rightarrow\; \frac{Q_\mathrm{p}}{g^{0.5} L_\mathrm{p}^{2.5}} = \frac{Q_\mathrm{m}}{g^{0.5} L_\mathrm{m}^{2.5}} \;\Rightarrow\; \frac{Q_\mathrm{m}}{Q_\mathrm{p}} = \left(\frac{L_\mathrm{m}}{L_\mathrm{p}}\right)^{2.5} = \varepsilon^{2.5} \tag{2}$$

- If the model and prototype turbines operate at the same conditions

$$\Psi = \frac{\Delta p_\mathrm{m}}{\rho_\mathrm{air} \Omega_\mathrm{m}^2 d_\mathrm{m}^2} = \frac{\Delta p_\mathrm{p}}{\rho_\mathrm{air} \Omega_\mathrm{p}^2 d_\mathrm{p}^2} \tag{3}$$

$$\Phi = \frac{Q_\mathrm{m}}{\Omega_\mathrm{m} d_\mathrm{m}^3} = \frac{Q_\mathrm{p}}{\Omega_\mathrm{p} d_\mathrm{p}^3} \tag{4}$$

- Applying (1) and (2) we get on (3) and (4)

$$\frac{d_\mathrm{m}}{d_\mathrm{p}} = \left(\frac{L_\mathrm{m}}{L_\mathrm{p}}\right) = \varepsilon \tag{5}$$

$$\frac{\Omega_\mathrm{m}}{\Omega_\mathrm{p}} = \left(\frac{L_\mathrm{p}}{L_\mathrm{m}}\right)^{0.5} = \varepsilon^{-0.5} \tag{6}$$

# How to scale a turbine

- Assuming a test scale of 1:100
  - Turbine with $d_p = 1.2\,\mathrm{m}$ at prototype scale (1:1)

$$d_\mathrm{m} = d_\mathrm{p}\left(\frac{1}{100}\right) = 0.012\,\mathrm{m} \Rightarrow \textbf{too small to be manufactured}$$

  - For a typical rotational speed of 750 rpm at prototype scale

$$\Omega_\mathrm{m} = \Omega_\mathrm{p}(100)^{0.5} = 7500\,\mathrm{rpm}$$

  - Reynolds number of the turbine rotor at prototype scale

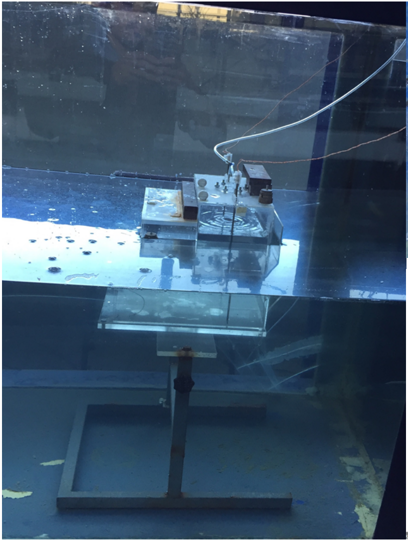
$$Re_\mathrm{p} = \frac{U_\mathrm{p}^\mathrm{tip} R_\mathrm{p}}{\nu} = \frac{\Omega_\mathrm{p} R_\mathrm{p}^2}{\nu} = 2 \times 10^6$$

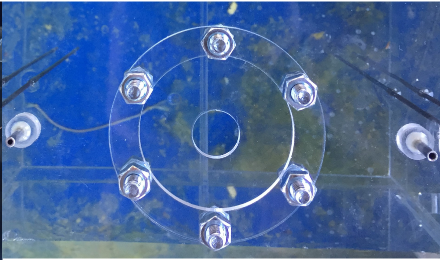
  - Reynolds number of the turbine rotor at model scale (1/100)

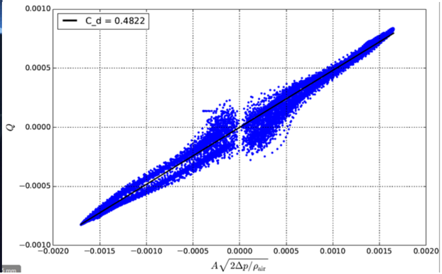
$$Re_\mathrm{m} = \frac{U_\mathrm{m}^\mathrm{tip} R_\mathrm{m}}{\nu} = \frac{\Omega_\mathrm{m} R_\mathrm{m}^2}{\nu} = 1.8 \times 10^4 \Rightarrow \textbf{too small to be meaningful}$$

  - Replace the turbine with a calibrated orifice/textile

# Simulation a quadratic turbine (axial impulse and biradial turbines)







# Simulation of a quadratic turbine (axial impulse and biradial turbines)

- For a quadratic turbine

$$\Phi = \kappa\sqrt{\Psi} \Rightarrow \frac{Q_\mathrm{m}}{\Omega_\mathrm{m} d_\mathrm{m}^3} = \kappa\sqrt{\frac{\Delta p_\mathrm{m}}{\rho_\mathrm{air}\Omega_\mathrm{m}^2 d_\mathrm{m}^2}} \Rightarrow Q_\mathrm{m} = \kappa d_\mathrm{m}^2\sqrt{\frac{\Delta p_\mathrm{m}}{\rho_\mathrm{air}}} \Rightarrow Q_\mathrm{m} = K_\mathrm{t}\sqrt{\frac{\Delta p_\mathrm{m}}{\rho_\mathrm{air}}}$$

- For an orifice ($Q_\mathrm{m} = u_\mathrm{o} A_\mathrm{o}$)

$$C_\mathrm{d} = \frac{\Delta p_\mathrm{m}}{\frac{1}{2}\rho_\mathrm{air} u_\mathrm{o}^2} \Rightarrow Q_\mathrm{m} = A_\mathrm{o}\sqrt{\frac{2}{C_\mathrm{d}}}\sqrt{\frac{\Delta p_\mathrm{m}}{\rho_\mathrm{air}}} \Rightarrow Q_\mathrm{m} = K_\mathrm{o}\sqrt{\frac{\Delta p_\mathrm{m}}{\rho_\mathrm{air}}}$$

- Typically $C_\mathrm{d} = 0.66$
- To simulate the turbine

$$K_\mathrm{o} = K_\mathrm{t}$$

- Since $A_\mathrm{o} = \pi d_\mathrm{o}^2/4$, we get

$$d_\mathrm{o} = C_\mathrm{d}^{0.25} 2^{0.75} \left(\frac{\kappa}{\pi}\right)^{0.5} d_\mathrm{m}$$

- The turbine diameter $d_\mathrm{m}$ scales using (5)

# Navier-Stokes equations and Reynolds number effects

- Navier-Stokes equations for laminar two-dimensional unsteady-flow
  - Continuity equation (conservation of mass)

    $$\nabla \cdot \mathbf{u} = 0$$

  - Conservation of momentum

    $$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \, \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} - \rho g \, \boldsymbol{e}_z$$

# Navier-Stokes equations and Reynolds number effects

- To scale the Navier-Stokes equations we need to find characteristic length, time, velocity and pressure of the system

- Possible choices (**not unique**!)
  - device diameter $[\,d\,] = L$
  - (phase) velocity $[\,U\,] = LT^{-1}$
  - density $[\,\rho\,] = ML^{-3}$

# Navier-Stokes equations and Reynolds number effects

- Since

$$t^* = \frac{U}{d}\, t \quad \text{and} \quad \mathbf{x}^* = \frac{1}{d}\, \mathbf{x},$$

the scaled time derivative and spatial derivatives are given by

$$\frac{\partial}{\partial t} = \frac{U}{d}\, \frac{\partial}{\partial \left( \frac{U}{d}\, t \right)} = \frac{U}{d}\, \frac{\partial}{\partial t^*}$$

$$\frac{\partial}{\partial x} = \frac{1}{d}\, \frac{\partial}{\partial \left( \frac{x}{d} \right)} = \frac{1}{d}\, \frac{\partial}{\partial x^*} \quad \Rightarrow \quad \nabla = \frac{1}{d}\, \nabla^*,$$

- The scaled continuity equation reads

$$\nabla^* \cdot \mathbf{u}^* = 0 \tag{7}$$

where

$$\mathbf{u} = U\, \mathbf{u}^*$$

# Navier-Stokes equations and Reynolds number effects

- To scale the momentum balance we need to define

$$p = \rho U^2 \, p^*$$

- Scaled conservation of momentum

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla^* p^* + \frac{1}{\mathrm{Re}} \nabla^{*2} u^* - \frac{1}{\mathrm{Fr}^2} \boldsymbol{e}_z \tag{8}$$

- Re is the **Reynolds** number

$$\mathrm{Re} = \frac{\rho U d}{\mu} \tag{9}$$

- Fr is the **Froude** number

$$\mathrm{Fr} = \frac{U}{\sqrt{gd}} \tag{10}$$

# Navier-Stokes equations and Reynolds number effects

- Scaled conservation of momentum

$$\underbrace{\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^*}_{F_i = \text{inertia terms}} = \underbrace{-\nabla^* p^*}_{F_{p=\text{pressure term}}} + \underbrace{\frac{1}{\text{Re}} \nabla^{*2} \mathbf{u}^*}_{F_v = \text{viscous terms}} - \underbrace{\frac{1}{\text{Fr}^2} \boldsymbol{e}_z}_{F_{p=\text{gravity term}}} \tag{11}$$

- Interpretation of the dimensionless numbers

$$\frac{F_i}{F_g} = \frac{\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^*}{\frac{1}{\text{Fr}^2} \boldsymbol{e}_z} \sim \text{Fr}^2 \tag{12}$$

$$\frac{F_i}{F_v} = \frac{\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^*}{\frac{1}{\text{Re}} \nabla^{*2} \mathbf{u}^*} \sim \text{Re} \tag{13}$$

# Navier-Stokes equations and Reynolds number effects

- Testing in similarity conditions implies that the Froude numbers of model (small-scale) and prototype (full-scale) must be the same

$$\frac{\mathrm{Fr_m}}{\mathrm{Fr_p}} = 1 \Leftrightarrow \left(\frac{U_\mathrm{m}}{\sqrt{g d_\mathrm{m}}}\right)\left(\frac{\sqrt{g d_\mathrm{p}}}{U_\mathrm{p}}\right) = 1 \Leftrightarrow \frac{U_m}{U_p} = \sqrt{\frac{d_\mathrm{m}}{d_\mathrm{p}}}$$

- Reynolds number do not scale as Froude

$$\frac{\mathrm{Re_m}}{\mathrm{Re_p}} = \left(\frac{U_\mathrm{m} d_\mathrm{m}}{\nu}\right)\left(\frac{\nu}{U_\mathrm{p} d_\mathrm{p}}\right) = \frac{U_m}{U_p}\frac{d_m}{d_p} = \sqrt{\frac{d_m}{d_p}}\frac{d_m}{d_p} < 1 \Rightarrow \mathrm{Re_m} < \mathrm{Re_p}$$

- **Can not scale both** Fr **and** Re

# Navier-Stokes equations and Reynolds number effects

- The effect of testing with different Reynolds number
  - Conservation of momentum

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla^* p^* + \underbrace{\frac{1}{\mathrm{Re}} \nabla^{*2} \mathbf{u}^*}_{\text{Not properly scaled}} - \frac{1}{\mathrm{Fr}^2} \boldsymbol{e}_z$$

- Since $\mathrm{Re_m} < \mathrm{Re_p}$
  - viscous effects are more important at model scale that at prototype scale
  - more damping $\Rightarrow$ smaller motion amplitudes $\Rightarrow$ smaller capture width

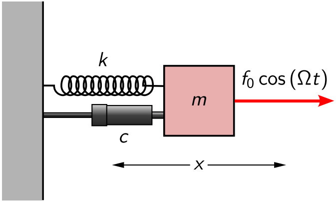
- **How the Reynolds number affects the response of a floating system excited by the waves?**

- **Why we perform decay tests?**

# Mass-spring-damper system dynamics

- **Let us study the effect of the damping in the dynamics of floating bodies**
- Consider a the dynamics of a mass-spring-damper system

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F\cos(\omega t)$$




- As such

$$x = f\left(t, m, c, k, F, \omega\right)$$

where

- $x$ - displacement $[L]$
- $t$ - time $[T]$
- $m$ - mass $[M]$
- $c$ - damping constant $[M/T]$
- $k$ - spring stiffness $\left[M/T^2\right]$
- $F$- driving force modulus $\left[ML/T^2\right]$
- $\omega$ - driving force frequency $[1/T]$

# Mass-spring-damper system dynamics

- Selecting a set of primary variables with the 3 dimensions $L, M, T$, i.e.,
  - $\{m,\ k,\ F\}$

  we get (see Buckingham Π theorem)

  $$x^* = f^*(t^*, c^*, \omega^*)$$

  where
  - $x^* = \dfrac{kx}{F}$ - dimensionless displacement
  - $t^* = t\sqrt{\dfrac{k}{m}}$ - dimensionless time
  - $c^* = \dfrac{c}{\sqrt{km}}$ - damping coefficient
  - $\omega^* = \omega\sqrt{\dfrac{m}{k}}$ - dimensionless frequency
- Hereinafter, **the $*$ does not denote the complex conjugate**

# Mass-spring-damper system dynamics

- From the dimensionless time

$$t^* = t\sqrt{\frac{k}{m}} \quad \Rightarrow \quad \frac{\mathrm{d}t^*}{\mathrm{d}t} = \sqrt{\frac{k}{m}}$$

  yields

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}t^*}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t^*} = \sqrt{\frac{k}{m}}\frac{\mathrm{d}}{\mathrm{d}t^*}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}t^*}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}t^{*2}} = \frac{k}{m}\frac{\mathrm{d}^2}{\mathrm{d}t^{*2}}$$

- Dividing the ODE by $F$ results

$$\frac{m}{F}\left(\frac{k}{m}\frac{\mathrm{d}^2 x}{\mathrm{d}t^{*2}}\right) + \frac{c}{F}\left(\sqrt{\frac{k}{m}}\frac{\mathrm{d}x}{\mathrm{d}t^*}\right) + x^* = \cos(\omega^* t^*)$$

$$\frac{\mathrm{d}^2 x^*}{\mathrm{d}t^{*2}} + c^*\frac{\mathrm{d}x^*}{\mathrm{d}t^*} + x^* = \cos(\omega^* t^*) \tag{14}$$

# Mass-spring-damper system dynamics

- The frequency domain solution

$$x^* = X\mathrm{e}^{\mathrm{i}\omega^* t^*}$$

implies that

$$\frac{\mathrm{d}x^*}{\mathrm{d}t^*} = \mathrm{i}\omega^* X\,\mathrm{e}^{\mathrm{i}\omega^* t^*} \quad \text{and} \quad \frac{\mathrm{d}^2x^*}{\mathrm{d}t^{*2}} = -\omega^{*2} X\,\mathrm{e}^{\mathrm{i}\omega^* t^*}$$

- Replacing in (14) and dividing $\mathrm{e}^{\mathrm{i}\omega^* t^*}$ by we get

$$\left(1 - \omega^{*2} + \mathrm{i}c^*\omega^*\right) X = 1$$

- Defining

$$R\mathrm{e}^{\mathrm{i}\theta^*} = 1 - \omega^{*2} + \mathrm{i}c^*\omega^*$$

results

$$X = \frac{1}{R}\,\mathrm{e}^{-\mathrm{i}\theta^*}$$

# Mass-spring-damper system dynamics

- Using the properties of the complex modulus we found

$$\left.\begin{aligned} |X| &= \frac{1}{|\,(1-\omega^{*2}) + \mathrm{i}c^*\omega^*|} = \frac{1}{R} \\ \angle X &= \arg(X) = -\theta^* \end{aligned}\right\} \quad \Rightarrow \quad x = \frac{F}{kR}\,\mathrm{e}^{\mathrm{i}\left(\omega^* t^* - \theta^*\right)}$$

where

$$R = \sqrt{(1-\omega^{*2})^2 + (c^*\omega^*)^2} \quad \text{and} \quad \tan\theta^* = \frac{c^*\omega^*}{1-\omega^{*2}}$$

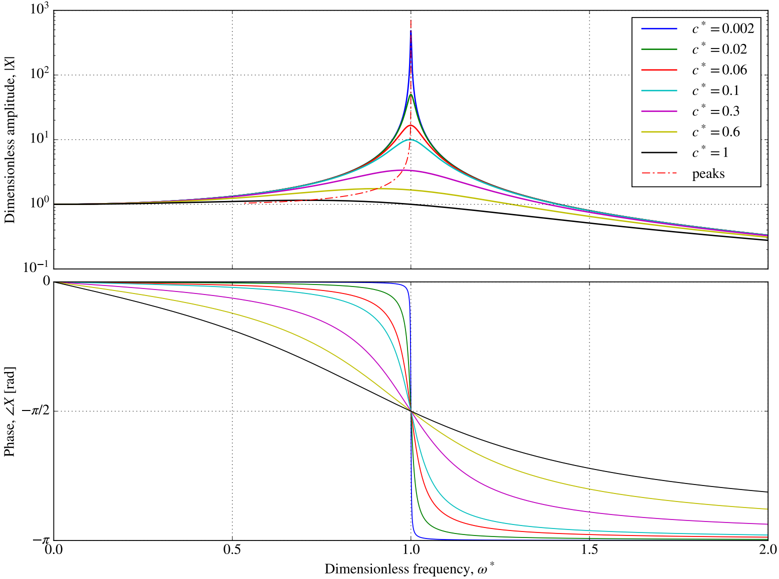
- Frequency at the maximum amplitude

$$\omega_{\text{peak}} = \sqrt{1 - \frac{c^{*2}}{2}}$$

- The natural frequency $\omega_0$ is given by $(c^* = 0)$

$$\omega_0^* = 1 \;\Leftrightarrow\; \omega_0\sqrt{\frac{m}{k}} = 1 \;\Leftrightarrow\; \omega_0 = \sqrt{\frac{k}{m}}$$

# Mass-spring-damper system dynamics

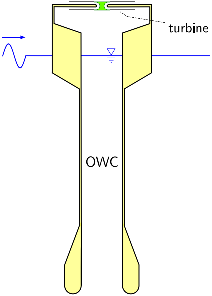


# Mass-spring-damper system dynamics

- Original system $x$ depends on **6 variables**: $t$, $m$, $c$, $k$, $F$ and $\omega$
- Dimensionless system $x^*$ depends on **3 variables**: $t^*$, $c^*$ and $\omega^*$
- The plot of the amplitude/phase response of the system depends only of **two variables** $c^*$ and $\omega^*$
- Two systems with the same $c^*$ are similar
- For the same $c^*$, the displacement is proportional to $F$

$$x = \frac{F}{kR} e^{i\left(\omega^* t^* - \theta^*\right)}$$

# Compressibility effects in an OWC air chamber




- Mass balance in the air chamber control volume

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}(\rho V)}{\mathrm{d}t} = \dot{\rho} V + \rho \dot{V} = -\dot{m}_t$$

$\dot{m}_t$ is the turbine mass flow rate (positive outward)

$$\frac{\dot{\rho}}{\rho} + \frac{\dot{V}}{V} = -\frac{\dot{m}_t}{\rho V} \tag{15}$$

- Assuming an pressure/density evolution of the air the chamber modelled as perfect gas

$$\frac{p + p_{\mathrm{atm}}}{\rho^{\gamma}} = \mathrm{const} \tag{16}$$

where

$$\begin{cases} k = 1, & \text{isotermic } (T = \text{const}) \\ 1 < k < 1.4, & \text{polytropic} \\ k = 1.4, & \text{isentropic (lossless)} \end{cases}$$

# Compressibility effects in an OWC air chamber

- Taking the logarithm of (16) we get

$$\log\left(p + p_{\text{atm}}\right) - k \log\left(\rho\right) = \text{const}$$

resulting

$$\frac{\dot{p}}{p + p_{\text{atm}}} - k\frac{\dot{\rho}}{\rho} = 0 \;\Rightarrow\; \frac{\dot{\rho}}{\rho} = \frac{\dot{p}}{k\left(p + p_{\text{atm}}\right)}$$

- Replacing in (15) we get

$$\frac{\dot{p}}{p + p_{\text{atm}}} = -k\left(\frac{\dot{V}}{V} + \frac{\dot{m}_t}{\rho V}\right) \tag{17}$$

- Eq. (17) show that we must use the **absolute pressure values**, $p + p_{\text{atm}}$
- The **instantaneous volume of the air chamber**, $V$, can greatly affect the pressure changes

# Compressibility effects in an OWC air chamber

- To have similarity between the model scale, $m$, and the full scale, $F$, the ratio between the LHS and the RHS of (17) must be equal for the two scales

$$\frac{\dfrac{\dot{p}_m}{p_m + p_{\mathrm{atm},m}}}{k_m\left(\dfrac{\dot{V}_m}{V_m} + \dfrac{\dot{m}_{t,m}}{\rho_m V_m}\right)} = \frac{\dfrac{\dot{p}_F}{p_F + p_{\mathrm{atm},F}}}{k_F\left(\dfrac{\dot{V}_F}{V_F} + \dfrac{\dot{m}_{t,F}}{\rho_F V_F}\right)}$$

- Typically $p \ll p_{\mathrm{atm}}$ and considering the stiffer case where $\dot{m}_t = 0$, we may approximate

$$\frac{\dot{p}_m}{\dot{p}_F} = \frac{k_m}{k_F}\frac{\dot{V}_m}{\dot{V}_F}\frac{V_F}{V_m}$$

- Since $\dot{V} = A\dot{x}$ we found that

$$\frac{V_m}{V_F} = \frac{k_m}{k_F}\frac{A_m}{A_F}\frac{\dot{x}_m}{\dot{x}_F}\frac{\dot{p}_F}{\dot{p}_m} \qquad (18)$$

where $\dot{x}$ is the OWC free surface velocity

# Compressibility effects in an OWC air chamber

- Using Froude scale, we get for each term of (18)

$$\frac{k_m}{k_F} = \frac{1}{k_F} \;\Rightarrow\; \text{for small scales, say } \varepsilon < \frac{1}{8}, \text{ we have isothermal flow } k_m = 1$$

$$\frac{A_m}{A_F} = \varepsilon^2$$

$$\frac{\dot{x}_m}{\dot{x}_F} = \varepsilon^{1/2}$$

$$\frac{\dot{p}_F}{\dot{p}_m} = \delta^{-1}\varepsilon^{-1/2}$$

where $\delta = \rho_m/\rho_F$ is the density ratio and $\varepsilon = L_m/L_p$ is the geometric scale

- Replacing in (18)

$$\frac{V_m}{V_F} = \frac{1}{k_F}\delta^{-1}\varepsilon^2 \qquad (19)$$

- **So, the volume should not scale with $\varepsilon^3$ as one may expect!**

# Compressibility effects in an OWC air chamber

- How to properly simulate the air compressibility

- Not practical to apply for floating OWCs!